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AN EXACT SOLUTION OF THE PERIODIC CONTACT PROBLEM FOR AN ELASTIC LAYER TAKING WEAR INTO ACCOUNT†

V. M. ALEKSANDROV and F. V. KUDROVA

Moscow

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Contact problems, when wear is taken into account, are usually reduced to distinctive integral equations containing a Fredholm operator with respect to the space coordinate and a Volterra operator with respect to time [1, 2]. In developing previous results [3] an effective method of solving such equations is developed for the case when the wear resistance on the surface of one of the interacting bodies changes periodically with respect to the space variable (the two-dimensional case). A similar problem was considered previously in a somewhat different formulation in [4, 5].

1. FORMULATION OF THE PROBLEM AND THE DERIVATION OF THE FUNDAMENTAL INTEGRAL EQUATION

Suppose an elastic layer of thickness h is rigidly attached to a base, and a rigid infinite plate is impressed into its upper face by a specific force $q(t)$, which has the dimensions of stress and which depends on the time t. The plate moves with constant velocity V in the direction of the z axis (Fig. 1), and forces of Coulomb friction arise in the region of contact between the plate and the layer, producing wear of the layer surface.

We will assume that the wear resistance of the surface of the layer varies periodically along the x axis in steps of $2l$. Such a surface structure may be formed, for example, by laser processing or by grinding with abrasives.

We will assume that abrasive wear occurs. Then, as has been established experimentally [6], the rate of linear wear

$$w = Vm(x)\tau(x, t) \tag{1.1}$$

where $m(x)$ is the wear resistance (a periodic function with period $2l$), and $\tau(x, t)$ is the shear contact force. Hence it follows that when the surface of the rigid plate is displaced in the direction of the y axis, we will have, due to wear of the layer,

$$v_* = -Vkm(x)\int_0^t q(x,\tau)d\tau \tag{1.2}$$

where k is the Coulomb friction coefficient and $q(x, t)$ is the contact pressure. Without loss of generality, we will henceforth use the following expression for the function $m(x)$

$$m(x) = m_0 + m_1\cos(\pi x/l), \quad m_0 > m_1 > 0 \tag{1.3}$$

The displacement of the surface of the rigid plate in the direction of the y axis, due to elastic deformation of the layer, is given by the formulae [7]

$$\begin{gathered} v = -\frac{1}{\pi\theta}\int_{-\infty}^{\infty} q(\xi,t)K\left(\frac{\xi-x}{h}\right)d\xi \\ \theta = \frac{G}{1-\nu}, \quad K(t) = \int_0^\infty \frac{L(u)}{u}\cos ut\,du \\ L(u) = \frac{2\varkappa\,\mathrm{sh}\,2u - 4u}{2\varkappa\,\mathrm{ch}\,2u + 1 + \varkappa^2 + 4u^2}, \quad \varkappa = 3 - 4\nu \end{gathered} \tag{1.4}$$

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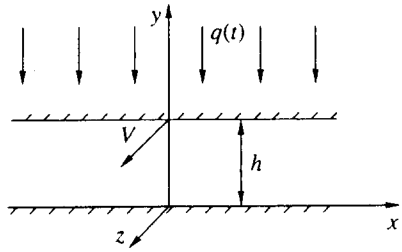

Fig. 1

Here G and ν are the elasticity constants of the layer and it is assumed that the antiplane deformation of the layer, due to the shear forces $\tau(x, t)$, is unrelated to its plane deformation.

The condition for the plate to be in contact with the layer when $y = h$ has the form

$$v_* + v = -\delta(t), \quad |x| < \infty \tag{1.5}$$

where $\delta(t)$ is the rigid displacement of the plate in the negative direction of the y axis due to the action of the force $q(t)$, where

$$q(t) = \frac{1}{2l}\int_{-l}^{l} q(\xi, t)d\xi \tag{1.6}$$

Substituting expressions (1.2) and (1.4) into (1.5), to determine the contact pressure we obtain the integral equation

$$\frac{1}{\pi\theta}\int_{-\infty}^{\infty} q(\xi, t)K\left(\frac{\xi - x}{h}\right)d\xi + Vkm(x)\int_0^t q(x, \tau)d\tau = \delta(t) \tag{1.7}$$

the solution of which must be obtained with the integral condition (1.6) in the time interval $0 \leqslant t \leqslant T$, where T is limited solely by the conditions for the displacements υ_* and υ to be comparable and for the pressure $q(x, t)$ to be non-negative for all $|x| < \infty$.

2. REDUCTION TO A SYSTEM OF SEQUENTIALLY SOLVED INTEGRAL EQUATIONS

In formulae (1.6) and (1.7) we will change to dimensionless quantities

$$\begin{gathered} x' = \frac{x}{l}, \quad t' = \frac{Vkm_0\theta t}{l}, \quad \lambda = \frac{h}{l}, \quad p(t') = \frac{q(t)}{\theta}, \quad \varphi(x', t') = \frac{q(x, t)}{\theta} \\ f(t') = \frac{\delta(t)}{l}, \quad n(x') = \frac{m(x)}{m_0} = 1 + m\cos\pi x', \quad m = \frac{m_1}{m_0} \end{gathered} \tag{2.1}$$

Omitting the primes, we will have

$$\begin{gathered} \frac{1}{\pi}\int_{-\infty}^{\infty} \varphi(\xi, t)K\left(\frac{\xi - x}{\lambda}\right)d\xi + n(x)\int_0^t \varphi(x, \tau)d\tau = f(t) \\ p(t) = \frac{1}{2}\int_{-1}^{1} \varphi(\xi, t)d\xi \end{gathered} \tag{2.2}$$

Suppose, further, that the change with time of the dimensionless rigid displacement of the plate $f(t)$ is specified. Following the well-known procedure [8], we will represent the functions $f(t)$ and $\varphi(x, t)$ in the form of a power series with respect to the small time parameter $\eta \in [0, 1]$

$$f(t) = \sum_{i=0}^{\infty} f_i\eta^i, \quad \varphi(x, t) = \sum_{i=0}^{\infty} \varphi_i(x)\eta^i; \quad \eta = 1 - e^{-\mu t} \tag{2.3}$$

and note that μ is an arbitrary constant; the assignment of a value to this depends on what range of variation of the time t it is desired to investigate.

We substitute series (2.3) into Eq. (2.2) and, taking into account the fact that

$$\int_0^t (1-e^{-\mu\tau})^i d\tau = \frac{1}{\mu}\sum_{s=i+1}^{\infty}\frac{1}{s}\eta^s \tag{2.4}$$

we obtain the relation

$$\frac{1}{\pi}\sum_{i=0}^{\infty}\eta^i\int_{-\infty}^{\infty}\varphi_i(\xi)K\left(\frac{\xi-x}{\lambda}\right)d\xi+\frac{n(x)}{\mu}\sum_{s=1}^{\infty}\frac{1}{s}\eta^s\sum_{i=0}^{s-1}\varphi_i(x)=\sum_{i=0}^{\infty}f_i\eta^i \tag{2.5}$$

Equating terms of like powers of η on the right and left, we arrive at an infinite system of integral equations for the sequential determination of the functions $\varphi_i(x)$

$$\begin{aligned}&\frac{1}{\pi}\int_{-\infty}^{\infty}\varphi_0(\xi)K\left(\frac{\xi-x}{\lambda}\right)d\xi=f_0\\&\frac{1}{\pi}\int_{-\infty}^{\infty}\varphi_i(\xi)K\left(\frac{\xi-x}{\lambda}\right)d\xi+\frac{n(x)}{i\mu}\sum_{s=0}^{i-1}\varphi_s(x)=f_i,\quad i\geqslant 1\end{aligned} \tag{2.6}$$

We will further show that any of Eqs (2.6) can be solved in closed form.

3. THE SOLUTION FOR THE CASE WHEN THE RELATIVE DISPLACEMENT OF THE PLATE IS SPECIFIED

We will seek solutions of Eqs (2.6) in the form

$$\varphi_i(x)=\sum_{j=0}^{i}A_j^{(i)}\cos\pi jx,\quad i\geqslant 0 \tag{3.1}$$

Taking representation (1.4) of the kernel $K(t)$ and the convolution theorem for the Fourier integral transform [9] into account, we obtain a solution of the first equation of (2.6) in the form

$$\varphi_0(x)=A_0^{(0)},\quad A_0^{(0)}=\frac{f_0}{\alpha_0},\quad \alpha_0=\frac{4(\varkappa-1)}{(\varkappa+1)^2} \tag{3.2}$$

We will put

$$F_i=f_i-\frac{1}{i\mu}(1+m\cos\pi x)\sum_{s=0}^{i-1}\varphi_s(x) \tag{3.3}$$

Substituting expression (3.1) into (3.3) and transforming, we obtain

$$\begin{aligned}&F_i=f_i-\frac{1}{i\mu}\left[\sum_{s=0}^{i-1}a_{i,s}\cos\pi sx+\frac{m}{2}\left(\sum_{s=0}^{i-2}a_{i,s+1}\cos\pi sx+\sum_{s=0}^{i}a_{i,s-1}\cos\pi sx\right)+ma_{i,0}\cos\pi ix\right];\\&a_{i,s}=\sum_{r=s}^{i-1}A_s^{(r)},\quad a_{i,s}=0,\quad s\geqslant i\end{aligned} \tag{3.4}$$

We will put

$$G_i=\frac{1}{\pi}\int_{-\infty}^{\infty}\varphi_i(\xi)K\left(\frac{\xi-x}{\lambda}\right)d\xi \tag{3.5}$$

Substituting expression (3.1) and expression (1.4) for the kernel $K(t)$ into (3.5) and again applying the convolution theorem, we obtain

$$G_i=\sum_{j=0}^{i}\alpha_jA_j^{(i)}\cos\pi jx,\quad \alpha_j=\frac{L(\pi j\lambda)}{\pi j},\quad j\geqslant 1 \tag{3.6}$$

Equating F_i of the form (3.4) and G_i of the form (3.6), by virtue of system (2.6), we obtain, when $i \geqslant 1$, the following recurrence system of relations for determining the coefficients $A_j^{(i)}$ in (3.1)

$$\begin{aligned}
&j=0: \quad \alpha_0 A_0^{(i)} = f_i - \frac{1}{i\mu}\left(a_{i,0} + \frac{m}{2}a_{i,1}\right)\\
&j=1: \ \alpha_1 A_1^{(i)} = -\frac{1}{i\mu}\left(a_{i,1} + \frac{m}{2}a_{i,2} + m a_{i,0}\right)\\
&2 \leqslant j \leqslant i-2: \ \alpha_j A_j^{(i)} = -\frac{1}{i\mu}\left[a_{i,j} + \frac{m}{2}(a_{i,j+1} + a_{i,j-1})\right]\\
&j=i-1: \ \alpha_{i-1} A_{i-1}^{(i)} = -\frac{1}{i\mu}\left(a_{i,i-1} + \frac{m}{2}a_{i,i-2}\right)\\
&j=i: \ \alpha_i A_i^{(i)} = -\frac{1}{i\mu} m a_{i,i-1}
\end{aligned} \tag{3.7}$$

Note that, when $i = 1$, one must only use the first two relations of (3.7), when $i = 2$ one must use the first two and the last relation, when $i = 3$ one must use the first two and the last two relations, and when $i \geqslant 4$ one must use all of relations (3.7).

Thus, as a result of solving integral equation (2.2) the required expression for the relative contact pressure $\varphi(x, t)$ is obtained. This is a power series in the reduced time and a trigonometric series in the coordinate.

Graphs were drawn of the relative rigid displacement $f(t)$ and the relative force $p(t)$ as a function of the time t (the continuous curves in Fig. 2), and also graphs of the relative contact pressure $\varphi(x, t)$ as a function of the x coordinate and the time t (Fig. 3 for the time interval (0, 1) and Fig. 4 for the time interval (0, 3)) for the following conditions: a relative thickness of the layer $\lambda = 2$, Poisson's ratio $\nu = 0.3$, a reduced time constant $\mu = 1.2$, a friction coefficient $k = 0.3$ and a wear resistance coefficient $m = 0.3$. In Fig. 5 (for the time interval (0,3)) for the above values of the parameters we show a graph of the worn surface of the layer, given by the expression

$$g(x,t) = \frac{n(x)}{\mu}\sum_{s=1}^{\infty}\frac{1}{s}\eta^s\sum_{i=0}^{s-1}\varphi_i(x) \tag{3.8}$$

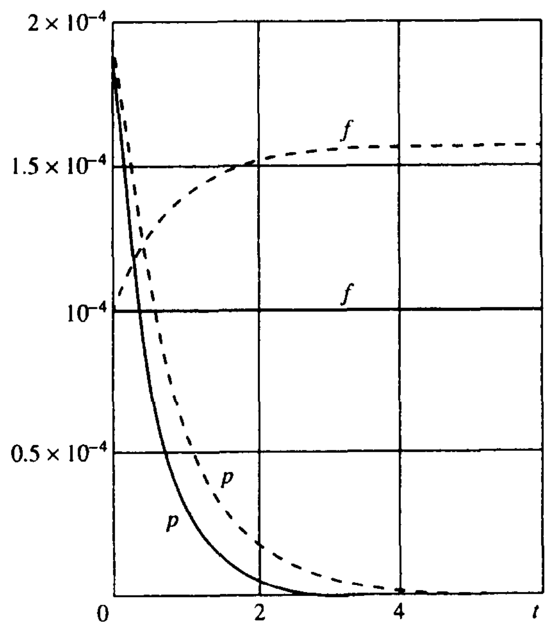

Fig. 2

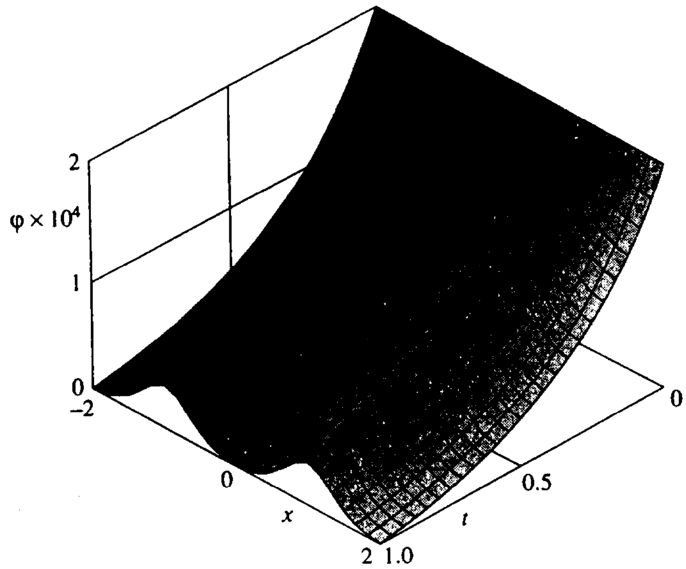

Fig. 3

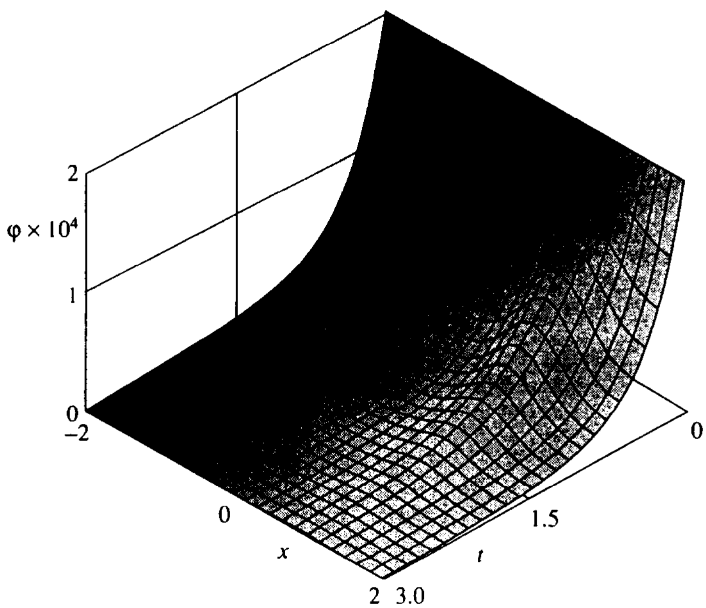

Fig. 4

as a function of the x coordinate and the time t. In this problem the relative rigid displacement $f(t)$ = 0.0001 was the initial displacement.

It can be seen from Fig. 2 that adjustment of the surfaces is completed when the dimensionless time $t \approx 4$.

The results obtained agree well with the physical picture of what occurs. In fact, suppose we are given the relative rigid displacement $f(t)$ = const. Then, at the initial instant, all points of the surface of the layer will experience a certain equal pressure, and then, after a certain time, as the surface of the layer wears away the pressure will weaken (this decrease will be non-uniform, since it depends on the variable wear resistance of the surface, i.e. it is periodic in this case) until the wear is such that the contacting surfaces are no longer in contact (the pressure at points of the layer surface vanishes). The relative force $p(t)$ in this case will tend exponentially to zero.

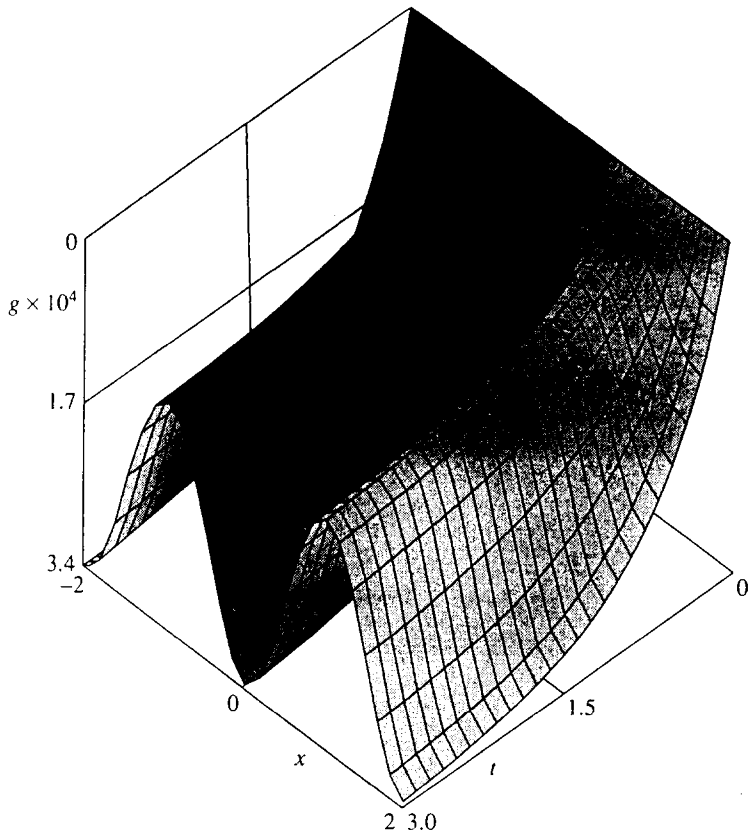

Fig. 5

4. THE SOLUTION FOR THE CASE WHEN THE RELATIVE FORCE ON THE PLATE IS SPECIFIED

Suppose now that we are given the change with time of the dimensionless specific force $p(t)$ rather than the change with time of the dimensionless rigid displacement of the plate $f(t)$ (this is simpler to realise in practice).

We expand the function $p(t)$ in series

$$p(t) = \sum_{i=0}^{\infty} p_i \eta^i \tag{4.1}$$

Substituting series (4.1) and the second series of (2.3) into the second relation of (2.2), we obtain

$$p_i = \frac{1}{2} \int_{-1}^{1} \varphi_i(\xi) d\xi \tag{4.2}$$

Now, taking (3.1) into account we find from (4.2) that

$$\varphi_0^{(i)} = p_i \tag{4.3}$$

and, consequently, by virtue of solution (3.2)

$$f_0 = \alpha_0 p_0 \tag{4.4}$$

Further, using recurrence formulae (3.7) and taking Eq. (4.3) into account we obtain successively $A_j^i (j \geq 1)$ and f_i.

For conditions similar to those indicated in Section 3, we drew graphs of the relative rigid displacement $f(t)$ and the relative force $p(t)$ as a function of the time t (the dashed curves in Fig. 2). The graphs of the relative contact pressure $\varphi(x, t)$, and also graphs of the worn surface of the layer $g(x, t)$, given by formula (3.8), as a function of the x coordinate and the time t are very similar in form to the corresponding graphs shown in Figs 3–5, and are therefore omitted.

Here the relative force $p(t) = 0.0002e^{-\mu t}$ was taken as the initial force.

It can be seen from Fig. 2 that in this case adjustment of the surfaces is completed when the dimensionless time $t \approx 6$.

The results obtained also agree well with the physical picture of what is occurring. In fact, if we specify the relative force, which tends exponentially to zero with time, then obviously this must lead to the fact that the relative rigid displacement $f(t)$ will tend to a certain constant value, and for the relative contact pressure we obtain relations similar to those in the direct problem.

5. THE ASYMPTOTIC SOLUTION FOR LONG TIMES

Consider once again the case of a relatively long time t, when

$$f(t) = f_\infty(t + \text{const}) + f_*(t) \tag{5.1}$$

where $f_*(t) \to 0$ monotonically as $t \to \infty$. In this case, by a simple analysis of Eq. (2.2) it can be shown that

$$\varphi(x,t) = f_\infty / n(x) + \varphi_*(x,t)$$

$$p(t) = f_\infty /(1 - m^2) + p_*(t) \tag{5.2}$$

where $\varphi_*(x, t) \to 0$ and $p_*(t) \to 0$ monotonically as $t \to \infty$. The inverse assertion also holds: if $p(t)$ has the form (5.2), then $f(t)$ is given by formula (5.1) and $\varphi(x, t)$ is given by the first formula of (5.2).

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REFERENCES

1. ALEKSANDROV, V. M., GALIN, L. A. and PIRIYEV, N. P., The plane contact problem with wear for a very thick elastic layer. *Izv. Akad. Nauk SSSR. MTT.*, 1978, 4, 60–67.
2. ALEKSANDROV, V. M. and KOVALENKO, Ye. V., Mathematical methods in contact problems with wear. In *Non-linear Models and Problems of the Mechanics of a Deformable Solid. Nauka*, Moscow, 1984, 77–89.
3. ALEKSANDROV, V. M., The periodic contact problem for an elastic layer taking friction and wear into account. In *Current Problems in Continuum Mechanics*. MP Kniga, Rostov-on-Don, 1995, 14–18.
4. GORYACHEVA, I. G. and DOBYCHIN, M. N., Wear of non-uniformly hardened surfaces. *Treniye i Iznos*, 1986, 7, 6, 985–992.
5. GORYACHEVA, I. G. and DOBYCHIN, M.N., *Contact Problems in Tribology*. Mashinostroyeniye, Moscow, 1988.
6. KRUSHCHEV, M. M. and BABICHEV, M. A., *Abrasive Wear*. Nauka, Moscow, 1970.
7. VOROVICH, I. I., ALEKSANDROV, V. M. and BABESHKO, V. A., *Non-classical Mixed Problems of the Theory of Elasticity*. Nauka, Moscow, 1974.
8. ALEKSANDROV, V. M., Contact problems in tribology. In *Mechanics and Scientific-Technical Progress*, Vol. 3, *Mechanics of a Deformable Solid*, Nauka, Moscow, 1988, 170–180.
9. BOCHNER, S., *Lectures on Fourier Integrals*. Princeton University Press, Princeton, 1959.

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